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COMPANIONSHIP AND KNOT GROUP REPRESENTATIONS

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A companionship argument is used to give a constructive geometric proof of a key result concerning the knot homomorph problem: Given elements μ and λ in a group G, is there a knot K in S^3 and a surjective representation $\rho: \pi_1(S^3 - K) \to G$, such that $\rho(m) = \mu$ and $\rho(l) = \lambda$, where m and l are the meridian and longitude of K. The result presented here is that if for some μ that normally generates G, the pair (μ, μ^n) is realizable, where n is the order of $H_1(G; Z)$, then the pair (ν, ν^n) is realizable for any normal generator ν.

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1. Introduction

A pair of elements, (μ, λ), in a group G is called *realizable* if there is an oriented knot K in S^3, and a surjective representation $\rho: \pi_1(S^3 - K) \to G$, such that $\rho(m) = \mu$ and $\rho(l) = \lambda$, where m and l are the meridian and longitude of K. In [5] the problem of determining which (μ, λ) are realizable is essentially reduced to purely algebraic conditions on μ, λ, and G. The conditions involve both $H_2(G; Z)$ and $H_3(G; Z)$. An important corollary of the general results, first presented in [4], is a condition on G that eliminates the need to analyze $H_3(G; Z)$. For instance, in the case of the symmetric group S_n, the corollary eliminates the long case by case reductions that occupy [1]. This is illustrated in Section 2 below. It is the purpose of this paper to present a direct geometric proof of that corollary.

In order to be realizable, a basic necessary condition is that μ *normally* generate G, and that G be *finitely generated.* Given this, any λ for which (μ, λ) is realizable must satisfy three further conditions:

(i) $\lambda \in G''$,
(ii) $[\mu, \lambda] = 1$,
(iii) $\langle \mu, \lambda \rangle = 0 \in H_2(G; Z)$.

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Here G'' denotes the second commutator subgroup of G, $[\mu, \lambda] = \mu\lambda\mu^{-1}\lambda^{-1}$, and $\langle\mu, \lambda\rangle$ is the Pontryagin product. See [1] for further definitions and a short proof of the necessity of these conditions.

It is proved in [5] that, in general, the conditions above do not imply realizability. However, it is shown by a fairly simple geometric argument that if (μ, μ^n) is realizable, where n is the order of the cyclic group $H_1(G; Z)$, then (μ, λ) is realizable whenever it meets the conditions above. The corollary of the more general results of [4] and [5], mentioned above, is that if (μ, μ^n) is realizable for some normal generator of G, it is realizable for every normal generator of G. This independence of the choice of μ appears as a somewhat mysterious consequence of deeper results in the earlier papers. The proof below is constructive, and illustrates why the result holds.

2. Summary of results

In this section a list of results is presented. Throughout this and the next section G will denote a finitely generated group with normal generator μ.

Theorem 1. *(μ, λ) is realizable for some $\lambda \in G$.*

This result was first proved in [2] in answer to a problem posed by Neuwirth [6, question U]. A simple constructive proof is given in [3].

Theorem 2. *If (μ, λ_1) and (μ, λ_2) are realizable, then so are (μ, λ_1^{-1})* and $(\mu, \lambda_1\lambda_2)$.

This is proved using the inverse of a knot and connected sums.

Theorem 3. *If λ satisfies the three conditions listed in the introduction, then $(\mu, \mu^{an}\lambda)$ is realizable for some integer a.*

Here is an outline of the proof of this given in [5]. The Pontryagin product condition permits the construction of a knot K in some 3-manifold M^3 and a representation $\rho : \pi_1(M^3 - K) \to G$ with $\rho(m) = \mu$ and $\rho(l) = \lambda$, where m and l are the meridian and longitude of K. Surgery can be performed on $M^3 - K$, over ρ, to produce a classical knot complement, $S^3 - K'$, with representation ρ'. It follows that $\rho(m') = \mu$ and $\rho(l') = \mu^b\lambda$. Since $\rho(l') \in G'$, b must be divisible by n.

Theorem 4. *(μ, μ^{n^2}) is realizable.*

In [5] this is proved as follows. Using Theorems 1 and 2, construct a knot K realizing $(\mu, 1)$. Find an unknotted circle in the complement of K that links K algebraically n times and that represents to 1. Performing +1 surgery on that unknotted circle yields the desired knot and representation.

Proposition 1. *If* (μ, μ^n) *is realizable, then* (ν, ν^n) *is realizable for every normal generator* ν *of* G.

This will be proved in the next section.

Corollary. *If* $G = S_k$ *and* (μ, λ) *satisfies the three conditions stated in the introduction, then* (μ, λ) *is realizable.*

This is the main result of [1]. Here is a proof. S_k is normally generated by a transposition, τ. By Theorems 1 and 2, (τ, τ^2) is realizable. Hence (μ, μ^2) is realizable for any normal generator. Now apply Theorem 3 to realize $(\mu, \mu^{2a}\lambda)$, and Theorem 2 eliminate the μ^{2a} factor.

3. Proof of Proposition 1

Proof. In the following argument, references to the choice of basepoints will be suppressed. Closed paths in spaces determine elements of the fundamental group of a space by joining the closed path to the basepoint with an arc. In the constructions below there is always a natural choice of that arc.

Write $\mu = \prod_{i=1}^{q} h_i \nu^{\varepsilon_i} h_i^{-1}$, with $\varepsilon_i = \pm 1$. Without loss of generality, we can assume that $h_1 = 1$ and that the set $\{h_i \nu^{\varepsilon_i} h_i^{-1}\}$ generates G.

Let F_q be the q-punctured disk, $B^2 - \{p_i\}$. There is a surjective representation $\rho_1 : \pi_1(F_q) \to G$ with $\rho_1(\gamma_i) = h_i \nu^{\varepsilon_i} h_i^{-1}$ and $\rho_1(\partial B^2) = \mu$, where the γ_i's are linking circles to the p_i.

Form the product $S^1 \times B^2$. Let L denote the link $S^1 \times \{p_i\}$. $\pi_1(S^1 \times B^2 - L) = Z \times \pi_1(F_q)$. Using the projection on the second factor we can define a representation $\rho_2 : \pi_1(S^1 \times B^2 - L) \to G$. The following properties are satisfied:

(i) $\rho_2(\{\text{pt.}\} \times \partial B^2) = \mu$;

(ii) $\rho_2(S^1 \times \{x\}) = 1$, where $x \in \partial B^2$;

(iii) $\rho_2(m') = \nu$, where m' is a meridian to the first component of L.

Set $\{\text{pt.}\} \times \partial B^2 = m$ and $S^1 \times \{x\} = l$. Note also that ρ_2 is surjective.

The next step is to make L connected. We will do this by using band connected sums. In general, suppose the meridian, m_1, of one component of a link satisfies $\rho(m_1) = g$, while the meridian, m_2, of another component satisfies $\rho(m_2) = hgh^{-1}$. A new link can be constructed by forming the band connected sum of the two components in such a way that there is still a representation to G carrying a meridian to g. In essence, the band is chosen so that ρ maps its core to h. (This depends on the surjectivity of ρ. The details are an exercise in keeping track of basepoints. Those details are presented in Proposition 4.7 of [1].

At this point we have constructed a knot $J \subseteq S^1 \times B^2$ and a surjective representation, denoted ρ, mapping $\pi_1(S^1 \times B^2 - J)$ to G such that $\rho(m) = \mu$, $\rho(l) = 1$, and

$\rho(m') = \nu$. Define a longitude, l', to J by picking a nontrivial simple closed curve on the peripheral torus to J that is homologous, in $S^1 \times B^2 - J$, to some multiple of $[l]$. Denote $\rho(l')$ by λ.

We now do two companionship arguments.

Let $K_1 \subseteq S^3$ be a knot such that there is a representation $\sigma_1 : \pi_1(S^3 - K_1) \to G$ for which $\sigma_1(m_1) = \mu$ and $\sigma_1(l_1) = 1$, where m_1 and l_1 are a meridian and longitude of K_1. Pick an embedding $f_1 : S^1 \times B^2 \to S^3$ such that $f_1(S^1 \times \{0\}) = K_1$ and $f_1(l) = l_1$. Set $J_1 = f_1(J)$. Note that ρ and σ together define a surjective representation $\phi_1 : \pi_1(S^3 - J_1) \to G$ with $\phi_1(f(m')) = \nu$ and $\phi_1(f(l')) = \lambda$. Note that $f(m')$ and $f(l')$ are a meridian and longitude of J_1. Denote them m_1' and l_1'.

Let $K_2 \subseteq S^3$ be a knot such that there is a surjective representation $\sigma_2 : \pi_1(S^3 - K_2) \to G$ such that $\sigma_2(m_2) = \mu$ and $\sigma_2(l_2) = \mu^n$, where m_2 and l_2 are a meridian and longitude of K_2. Pick an embedding $f_2 : S^1 \times B^2 \to S^3$ such that $f_2(S^1 \times \{0\}) = K_2$ and $f_2(l)$ is homotopic to $l_2 \cdot m_2^{-n}$ on the peripheral torus of K_2. (Note that $\rho(l) = 1 = \sigma_2(l_2 \cdot m_2^{-n})$.) Set $J_2 = f_2(J)$. In this case ρ and σ_2 can be used to define a representation $\phi_2 : \pi_1(S^3 - J_2) \to G$ such that $\phi_2(f_2(m')) = \nu$ and $\phi_2(f_2(l')) = \lambda$. As before, $f_2(m')$ is a meridian to J_2, which we denote m_2'. However, because of the twist coming from f_2, $f_2(l')$ does not represent a longitude to J_2. A straightforward calculation shows that $f_2(l')$ is homologous to $m_2'^{\,d^2 n}$ in $S^3 - J_2$, where d is the algebraic winding number of J in $S^1 \times B^2$. It follows that the longitude to J_2, l_2', is represented by a simple closed curve on the peripheral torus to J_2 which is homotopic to $f(l') \cdot m_2'^{\,-d^2 n}$. Hence, $\phi_2(m_2') = \nu$ and $\phi_2(l_2') = \lambda \nu^{-d^2 n}$.

Using Theorem 2 we see that since (ν, λ) and $(\nu, \lambda\nu^{-d^2 n})$ are both realizable, $(\nu, \nu^{-d^2 n})$ is realizable. We must now study d.

We have assumed that $[\mu] = 1 \in G/G' = Z_n$. As ν is a normal generator of G, $[\nu] = k \in G/G'$, where $(k, n) = 1$. Observe that $[m] = d[m']$ in $H_1(S^1 \times B^2 - J)$. It follows that $d \cdot k = 1 \bmod n$. This implies that d is relatively prime to n. Hence $(d^2, n) = 1$ as well. From this we conclude that there are integers, a and b, so that $ad^2 + bn = 1$.

By the argument just given, $(\nu, \nu^{d^2 n})$ is realizable. By Theorem 4, (ν, ν^{bn^2}) is realizable. Using Theorem 2 again, we find $(\nu, \nu^{ad^2 n + bn^2})$ is realizable. Finally, $ad^2 n + bn^2 = (ad^2 + bn)n = n$. Hence (ν, ν^n) is realizable. □

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